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APPROXIMATE SOLUTIONS OF A CLASS OF SIMILARITY EQUATIONS FOR THREE-DIMENSIONAL, LAMINAR, INCOMPRESSIBLE BOUNDARY-LAYER FLOWS

By Arthur G. Hansen and Howard Z. Herzig

Lewis Flight Propulsion Laboratory
Cleveland, Ohio

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SUMMARY

An analysis is presented for obtaining approximate solutions of the similarity equations for three-dimensional laminar-boundary-layer flows over a flat surface under main-flow streamlines that are translates and representable as infinite series expansions. For the particular case of streamline shapes described by a power of the distance along the surface from the leading edge, relatively simple expressions are obtained for flow deflection at the boundary surface, limiting streamline shape, and shear stress at the surface.

INTRODUCTION

In recent years, a great deal of attention has been focused on theoretical investigations of three-dimensional incompressible boundary-layer flows. One class of investigations, having application to internal flow problems in turbomachines and flow over wings at high altitudes, has been concerned with finding exact solutions of the boundary-layer equations when the boundary layer develops over a flat surface (e.g., refs. 1 to 8). To date, all exact solutions have been based on similarity-type boundary-layer analyses. The essence of this technique involves the reduction of the partial differential equations for the boundary layer to a system of ordinary differential equations. The solutions for the actual boundary-layer flow are then obtained from the solutions of the ordinary differential equations. It is generally necessary, however, to employ numerical methods and high-speed computing equipment to obtain accurate solutions to the ordinary differential equations because of their complex nature. As this process is time-consuming and often laborious, it is of interest to determine whether or not approximate solutions to such equations might be readily obtained which would encompass a wide variety of boundary-layer flows.

Certain steps in this direction are taken in reference 9. In particular, approximate solutions are obtained for the ordinary differential equations arising from the similarity solution of the boundary-layer equations corresponding to the mainstream flows over a flat plate characterized by

$$U = U_0 \tag{1a}$$

$$W = (\text{const.})\ x^n \tag{1b}$$

(All symbols are defined in appendix A. The orientation of the coordinate axes is shown in fig. 1). The method of solution involves finding an approximate solution of the equations corresponding to the region of boundary-layer flow near the plate surface and then finding a separate approximate solution in the region near the main stream. The two individual solutions are then joined at a suitable point between these two regions (see also ref. 10, which advocates this technique). Comparison of the approximate solutions of reference 9 with complete solutions obtained by relaxation techniques shows fairly good agreement when the value of n in equation (1b) is large (e.g., $n = 10$). This corresponds to the physical case of mainstream flows that flow for a distance without appreciable turning and then turn and accelerate rapidly, generating strong crossflows in the boundary layer.

The present investigation also considers approximate solutions of the boundary-layer equations for the case of mainstream flows defined by equations (1). The reasons are twofold. First, although the approximate solution follows the procedure (advocated in refs. 9 and 10) of finding separate solutions of governing equations near the plate surface and near the main stream, the method of solution for the flow region near the plate surface differs from that given in reference 9 and hence provides a technique of solution that is of interest in its own right. The method presented herein is designed to give a more accurate approximation in the region near the plate surface and should yield improved estimates of shear stress and flow deflection at the surface.

Secondly, reference 5 has shown that boundary-layer solutions for particular values of n in equation (1b) can be linearly combined to yield solutions of boundary-layer flows arising from mainstream flows given by

$$U = U_0 \tag{2a}$$

$$W = \sum_{n=0}^{m} a_n x^n \tag{2b}$$

Because of computing-machine and time limitations, reference 5 obtains solutions only for a range of values of n from 1 to 10. This restricts the analysis to consideration of main-flow streamline shapes that can be approximated by eleventh-degree polynomials in x. Provided accurate approximate solutions can be obtained for values of $n > 10$, the analysis of reference 5 could correspondingly be extended to include the calculation of boundary layers for main-flow streamline shapes requiring approximating polynomials of degree higher than 11. For example, such approximating polynomials would be required for circular-arc flows having more than 60° turning.

The results of the present investigation and the approximate solutions of reference 9 are both compared with the exact solutions of reference 5, and the nature of both approximate solutions is discussed.

The work contained in this report is also included as part of a larger study presented in a doctoral thesis (ref. 11). Another portion of this thesis is presented in reference 12.

ANALYSIS

The solution for the boundary-layer velocity components for main flows over a flat surface defined by equations (2) is known to be (ref. 5)

$$u = U_0 F'(\eta) \tag{3}$$

$$v = \frac{1}{2} \sqrt{\frac{\nu U_0}{x}} (\eta F' - F) \tag{4}$$

$$W = \sum_{n=0}^{m} a_n x^n P_n(\eta) \tag{5}$$

where $F(\eta)$ is the well-known Blasius function satisfying the equation

$$\frac{FF''}{2} + F''' = 0 \tag{6}$$

where

$$\eta = y \sqrt{\frac{U_0}{\nu x}}$$

and P_n satisfies the equation

$$P_n'' + \frac{FP_n'}{2} - nF'P_n + n = 0 \tag{7}$$

The boundary conditions on equations (6) and (7) are

$$F(0) = F'(0) = P_n(0) = 0 \tag{8}$$

and

$$\lim_{\eta\to\infty} F'(\eta) = \lim_{\eta\to\infty} P_n(\eta) = 1 \tag{9}$$

Solutions to equation (7) (flows described by eqs. (2)) have been obtained for $0 \leq n \leq 10$. Approximate solutions of equation (7) are now sought for all $n > 10$. To accomplish this, an approximate solution is first developed for large values of η (i.e., in the vicinity of the main stream). Solutions are then found for small values of η (i.e., in the vicinity of the bounding surface). These two solutions are then matched at a suitable value of η.

Approximate Solution for Large η

An approximate solution of equation (7) for large η and large values of n is discussed in reference 13 and will be used in the present investigation. Basically, the solution is obtained as follows: For large values of η, it can be shown that P_n'' and P_n' are necessarily small. Hence, if n is large, equation (7) can be written as $F'P_n \approx 1$; and, therefore, the approximate solution for large η and n can be given by

$$P_n = \frac{1}{F'} \tag{10}$$

Essentially the same result was obtained in reference 9 by considering the upper part of the boundary layer to be a "nonviscous" region and finding a solution of the boundary-layer equations with the viscosity term neglected. References 9 and 13 also point out that this approximate solution applies over progressively wider ranges of values of η for progressively larger values of n.

Approximate Solution for Small η

In order to obtain an approximate solution of equation (7) for small values of η, an adaptation of an analysis used in reference 14 to calculate skin-friction coefficients will be employed. In this regard, a new function $\mathscr{F}_n(\eta)$ is defined, where $\mathscr{F}_n$ is related to P_n by the equation

$$P_n \equiv F'\mathscr{F}_n \tag{11}$$

Substituting equation (11) into equation (7) and employing equation (6) give

$$\mathscr{F}_n''F' + \left(2F'' + \frac{FF'}{2}\right)\mathscr{F}_n' - n(F')^2\mathscr{F}_n = -n \tag{12}$$

From the series expansion for the Blasius function F (see ref. 15, p. 104) it is now assumed that F is adequately represented by

$$F = \frac{1}{2}\alpha\eta^2 \qquad (\alpha = F''(0) = 0.33206)$$

in the neighborhood of $\eta = 0$. Furthermore, examination of the second coefficient in equation (12) shows that $FF'/2$ is of the order of η^3 relative to $2F''$; and, hence, it is further assumed that in the neighborhood of $\eta = 0$ the coefficient can be replaced by $2F'' \approx 2\alpha$. Equation (12) can then be written as

$$\mathscr{F}_n'' + \frac{2\mathscr{F}_n'}{\eta} - n\alpha\eta\mathscr{F}_n = \frac{-n}{\alpha\eta} \tag{13}$$

At present, these approximations will be assumed sufficiently accurate for the purpose of the investigation. Later, results obtained from equation (13) are compared with exact solutions in order to establish the validity of the assumptions.

The solution of equation (13) can now be obtained analytically as follows. Let

$$\mathscr{F}_n = \zeta^{-\frac{1}{3}} H_n(\zeta) \tag{14a}$$

where

$$\zeta = \frac{2}{3} n^{\frac{1}{2}}\alpha^{\frac{1}{2}}\eta^{\frac{3}{2}} \tag{14b}$$

Substitution of equations (14) into equation (13) then yields

$$H_n'' + \frac{H_n'}{\zeta} - \left(\frac{1}{9}\zeta^2 + 1\right)H_n = -n^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}\alpha^{-\frac{4}{3}}\zeta^{-1} \tag{15}$$

Consider the homogeneous complementary equation formed from equation (15)

$$H_{n,c}'' + \frac{H_{n,c}'}{\zeta} - \left(\frac{1}{9}\zeta^2 + 1\right)H_{n,c} = 0 \tag{16}$$

Equation (16) is a form of Bessel's equation and has the general solution

$$H_{n,c} = A_n I_{\frac{1}{3}} + B_n K_{\frac{1}{3}} \tag{17}$$

where $I_{\frac{1}{3}}$ and $K_{\frac{1}{3}}$ are Bessel functions of imaginary arguments (see ref. 16 for definitions and discussion). Tables of these functions are given in reference 17.

If the solution $H_{n,c}$ of the complementary equation (16) is known, the solution of equation (15) can be obtained by the method of variation of parameters. An analysis using this method is presented in appendix B, and the result is

$$H_n = n^{\frac{2}{3}}\alpha^{-\frac{4}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}\left(K_{\frac{1}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - I_{\frac{1}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + A_n I_{\frac{1}{3}} + B_n K_{\frac{1}{3}}\right) \tag{18}$$

Hence, from equations (18), (14), and (11), the solution for P_n is

$$P_n(\eta) = F'(\eta)\ \zeta^{-\frac{1}{3}}n^{\frac{2}{3}}\alpha^{-\frac{4}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}\left(K_{\frac{1}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - I_{\frac{1}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + A_n I_{\frac{1}{3}} + B_n K_{\frac{1}{3}}\right) \tag{19}$$

It is now recalled that, for regions near the wall, $F(\eta)$ was assumed to be adequately represented by the first term of its series expansion; that is,

$$F = \frac{\alpha\eta^2}{2}$$

Using this approximation for F and using equation (14b), the expression for P_n given by equation (19) becomes

$$P_n(\zeta) = \alpha^{-\frac{2}{3}}n^{\frac{1}{3}}\left(\frac{3}{2}\right)^{-\frac{2}{3}}\zeta^{\frac{1}{3}}\left(K_{\frac{1}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - I_{\frac{1}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + A_n I_{\frac{1}{3}} + B_n K_{\frac{1}{3}}\right) \tag{20}$$

It is to be noted that equation (20) contains two arbitrary constants A_n and B_n. One of these constants can be determined by the boundary conditions $P_n(0) = 0$. Setting η equal to zero in equation (20) (equivalent to setting $\zeta = 0$) gives

$$P_n(0) = \frac{\pi\alpha^{-\frac{2}{3}}n^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{2}{3}} B_n}{\sqrt{3}\ \Gamma\left(\frac{2}{3}\right)}2^{\frac{1}{3}} = 0 \tag{21}$$

where the series expansions for $I_{\frac{1}{3}}(\zeta)$ and $K_{\frac{1}{3}}(\zeta)$ (ref. 16) are used in determining the equation. It follows at once from equation (21) that $B_n = 0$.

The determination of the remaining constant A_n depends upon the matching of the solutions for large and small η and is considered in the following section.

Matching of Solutions

The matching of the solutions for large and small η consists in finding a value of $\overline{\eta}$ where the values of P_n and P_n' for both cases can be brought into agreement. The equating of the functions and the derivatives gives a system of two simultaneous equations for the determination of $\overline{\eta}$ and the remaining arbitrary constant A_n.

The value of $dP_n/d\eta$ can be calculated from equation (20) by taking

$$\frac{dP_n}{d\eta} = \frac{dP_n}{d\zeta}\frac{d\zeta}{d\eta} = \frac{dP_n}{d\zeta} n^{\frac{1}{3}}\alpha^{\frac{1}{3}}\left(\frac{3}{2}\right)^{\frac{1}{3}} \zeta^{\frac{1}{3}} \tag{22}$$

Differentiating equation (20) with respect to ζ gives

$$\frac{dP_n}{d\zeta} = \alpha^{-\frac{2}{3}}n^{\frac{1}{3}}\left(\frac{3}{2}\right)^{-\frac{2}{3}} \zeta^{\frac{1}{3}} \left[\left(\frac{1}{3}\zeta^{-1} K_{\frac{1}{3}} + K_{\frac{1}{3}}'\right)\int_0^{\zeta} I_{\frac{1}{3}}\, d\zeta - \left(\frac{1}{3}\zeta^{-1} I_{\frac{1}{3}} + I_{\frac{1}{3}}'\right)\int_0^{\zeta} K_{\frac{1}{3}}\, d\zeta + A_n\left(\frac{1}{3}\zeta^{-1} I_{\frac{1}{3}} + I_{\frac{1}{3}}'\right)\right] \tag{23}$$

Now, from reference 16 (p. 79), the following relations are known:

$$\frac{1}{3}\zeta^{-1} K_{\frac{1}{3}} + K_{\frac{1}{3}}' = -K_{-\frac{2}{3}} \tag{24}$$

$$\frac{1}{3}\zeta^{-1} I_{\frac{1}{3}} + I_{\frac{1}{3}}' = I_{-\frac{2}{3}} \tag{25}$$

Successive substitution of equations (24) and (25) into equation (23), and equation (23) into equation (22), gives

$$\frac{dP_n}{d\eta} = -\alpha^{-\frac{1}{3}}n^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{1}{3}}\zeta^{\frac{2}{3}}\left(I_{-\frac{2}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + K_{-\frac{2}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - A_n I_{-\frac{2}{3}}\right) \tag{26}$$

The corresponding value of $dP_n/d\eta$ for the solution of the upper region of the boundary layer is determined from equation (10). Differentiating equation (10) with respect to η gives

$$\frac{dP_n}{d\eta} = -\frac{1}{(F')^2}F'' \tag{27}$$

At this point, it is assumed that the equations can be matched in a region where F is reasonably approximated by $F = \alpha\eta^2/2$. It will be shown later that this assumption will become increasingly accurate for larger values of n. Using this approximation for F and equating equations (10) and (20) and equations (26) and (27) give the following system of simultaneous equations for A_n and the matching point $\bar{\eta}$. (Actually, the equations are given as functions of ζ and a matching value $\bar{\zeta}$; $\bar{\eta}$ can then be computed by means of the relation between ζ and η given by eq. (14b).)

$$\alpha^{-\frac{2}{3}}n^{\frac{1}{3}}\left(\frac{3}{2}\right)^{-\frac{2}{3}}\zeta^{\frac{1}{3}}\left(K_{\frac{1}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - I_{\frac{1}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + A_n I_{\frac{1}{3}}\right)$$

$$= \frac{1}{F'} = \frac{1}{\alpha\eta} = \zeta^{-\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{2}{3}}n^{\frac{1}{3}}\alpha^{-\frac{2}{3}} \tag{28}$$

$$\alpha^{-\frac{1}{3}}n^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{1}{3}}\zeta^{\frac{2}{3}}\left(I_{-\frac{2}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta + K_{-\frac{2}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta - A_n I_{-\frac{2}{3}}\right)$$

$$= \frac{-F''}{(F')^2} = -\frac{1}{\alpha\eta^2} = -\zeta^{-\frac{4}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}n^{\frac{2}{3}}\alpha^{-\frac{1}{3}} \tag{29}$$

From equation (28) it follows that

$$A_n = \frac{\zeta^{-1} + I_{\frac{1}{3}}\int_0^\zeta K_{\frac{1}{3}}\,d\zeta - K_{\frac{1}{3}}\int_0^\zeta I_{\frac{1}{3}}\,d\zeta}{I_{\frac{1}{3}}} \tag{30}$$

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Substituting equation (30) into equation (29) gives

$$I_{-\frac{2}{3}}\int_0^{\zeta} K_{\frac{1}{3}}\,d\zeta + K_{-\frac{2}{3}}\int_0^{\zeta} I_{\frac{1}{3}}\,d\zeta - \frac{I_{-\frac{2}{3}}}{I_{\frac{1}{3}}}\left(\zeta^{-1} + I_{\frac{1}{3}}\int_0^{\zeta} K_{\frac{1}{3}}\,d\zeta - K_{\frac{1}{3}}\int_0^{\zeta} I_{\frac{1}{3}}\,d\zeta\right) = \frac{2}{3}\zeta^{-2} \tag{31}$$

The solution of equation (31), which gives the value of ζ at which the two solutions are matched, is outlined as follows: Multiplying equation (31) through by $I_{\frac{1}{3}}$ and simplifying give

$$\left(I_{\frac{1}{3}}K_{-\frac{2}{3}} + K_{\frac{1}{3}}I_{-\frac{2}{3}}\right)\cdot\int_0^{\zeta} I_{\frac{1}{3}}\,d\zeta - \zeta^{-1}\,I_{-\frac{2}{3}} = \frac{2}{3}\,\zeta^{-2}\,I_{\frac{1}{3}} \tag{32}$$

Now, $I_{\frac{1}{3}}K_{-\frac{2}{3}} + K_{\frac{1}{3}}I_{-\frac{2}{3}} = \zeta^{-1}$ (ref. 16, p. 80). Therefore, equation (32) becomes

$$\int_0^{\zeta} I_{\frac{1}{3}}\,d\zeta = I_{-\frac{2}{3}} + \frac{2}{3}\,\zeta^{-1}\,I_{\frac{1}{3}} \tag{33}$$

Numerical solution of equation (33) gives

$$\zeta = \overline{\zeta} = 2.96 \tag{34}$$

It follows that $\overline{\eta}$ is given by

$$\overline{\eta} = (\overline{\zeta})^{\frac{2}{3}}\left(\frac{3}{2}\right)^{\frac{2}{3}}\alpha^{-\frac{1}{3}}n^{-\frac{1}{3}}$$

$$\therefore\ \overline{\eta} = \frac{3.902}{n^{\frac{1}{3}}} \tag{35}$$

From the value of $\overline{\zeta}$ given in equation (34), it follows from equation (30) that the value of A_n is

$$A_n = 1.811 \tag{36}$$

Calculation of $P_n(\eta)$ Curves

From equations (20), (36), and the fact that $B_n \equiv 0$, $P_n(\eta)$ can be expressed

$$P_n(\eta) = n^{\frac{1}{3}}\, Q(\zeta) \tag{37}$$

where

$$Q(\zeta) = \alpha^{-\frac{2}{3}} \left(\frac{2}{3}\right)^{\frac{2}{3}} \zeta^{\frac{1}{3}} \left(K_{\frac{1}{3}} \int_0^{\zeta} I_{\frac{1}{3}}\, d\zeta - I_{\frac{1}{3}} \int_0^{\zeta} K_{\frac{1}{3}}\, d\zeta + 1.811\, I_{\frac{1}{3}}\right) \tag{38}$$

Figure 2 shows a plot of $Q(\zeta)$ over the range $0 \leqq \zeta \leqq \bar{\zeta} = 2.96$. In order to obtain a plot of $P_n(\eta)$ for a given value of n, it is necessary merely to select values of η, determine the corresponding values of ζ from equation (14b), find $Q(\zeta)$ from figure 2, and then determine $P_n(\eta)$ from equation (37).

A plot of the variation of η with ζ is presented in figure 3 for values of n equal to 10, 15, 20, and 50. Using these values of n and figures 2 and 3, curves for $P_n(\eta)$ were calculated and are presented in figure 4.

As a check on the validity of the approximate solution, values of $P_{10}(\eta)$ taken from reference 5 are plotted in figure 4. As the plot indicates, the values of the approximate solution fall slightly below the exact values for the "middle range" of η values but agree very well for large η and for η near zero.

Flow Behavior Near Bounding Surface

Values of shear stress and limiting flow deflection near the bounding surface are of major interest in the analysis of three-dimensional boundary-layer flows. For the present case, both of these quantities can be shown to depend upon $P_n'(0)$ (see ref. 5). In the following section, therefore, an expression for $P_n'(0)$ is developed from equation (26), and values obtained are compared with exact values from reference 5 and the approximate solution of reference 9.

Determination of $P_n'(0)$. - From reference 16, the following expressions are noted:

$$I_{-\frac{2}{3}} = \frac{\left(\frac{1}{2}\zeta\right)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \text{higher-order terms}$$

$$K_{\frac{1}{3}} = \frac{\pi}{\sqrt{3}} \frac{\left(\frac{1}{2}\zeta\right)^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} + \text{higher-order terms}$$

$$K_{-\frac{2}{3}} = \frac{\pi}{\sqrt{3}} \frac{\left(\frac{1}{2}\zeta\right)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \text{higher-order terms}$$

$$I_{\frac{1}{3}} = \frac{\left(\frac{1}{2}\zeta\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} + \text{higher-order terms}$$

Hence,

$$I_{-\frac{2}{3}} \int_0^{\zeta} K_{\frac{1}{3}}\, d\zeta = \frac{\sqrt{3}\,\pi}{4\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} + \text{higher-order terms} \tag{39}$$

$$K_{-\frac{2}{3}} \int_0^{\zeta} I_{\frac{1}{3}}\, d\zeta = \frac{\left(\frac{1}{2}\right)^{-\frac{1}{3}}\sqrt{3}\,\pi}{4\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)}\zeta^{\frac{2}{3}} + \text{higher-order terms} \tag{40}$$

Employing (39) and (40) in equation (26) gives

$$\frac{dP_n}{d\eta} = \alpha^{-\frac{1}{3}} n^{\frac{2}{3}} \left(\frac{3}{2}\right)^{-\frac{1}{3}} \left[\frac{A_n\left(\frac{1}{2}\right)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \text{higher-order terms}\right]$$

Therefore, at the wall where $\zeta = 0$,

$$\left(\frac{dP_n}{d\eta}\right)_{\eta=0} = 2\,\frac{\alpha^{-\frac{1}{3}} n^{\frac{2}{3}} A_n}{\Gamma\left(\frac{1}{3}\right)\sqrt[3]{3}} = 1.354\, n^{\frac{2}{3}} \tag{41}$$

The following table gives a comparison between the values calculated from equation (41) and the exact values computed in reference 5:

n	$P_n'(0)$, exact values from ref. 5	$P_n'(0)$, eq. (41)
1	1.418	1.354
2	2.197	2.149
3	2.858	2.816
4	3.450	3.412
5	3.995	3.959
6	4.506	4.470
7	4.989	4.954
8	5.449	5.416
9	5.891	5.858
10	6.317	6.284

It can be seen from the table that, for $n = 8$, 9, and 10, there is an error of approximately 1/2 percent between the approximate solution for $P_n'(0)$ and the values presented in reference 5.

At this point it is of interest to compare the values obtained for $P_n'(0)$ in the present analysis with those obtained by the approximation technique in reference 9. The method employed in the reference is based on expressing $P_n(\eta)$ as a cubic polynomial in η for small values of η and as $P_n = 1/F'(\eta)$ for large values. The two solutions are matched in a manner similar to that presented in the present investigation. However, as would be expected, the matching point $\bar{\eta}$ is not the same for the two analyses. In reference 9 the matching point is given by

$$\bar{\eta} = \left\{\frac{24}{\alpha}\left(1 + \frac{3}{2}n\right)\left[-1 + \sqrt{1 + \frac{2}{\left(1 + \frac{3}{2}n\right)^2}}\right]\right\}^{\frac{1}{3}} \tag{42}$$

It might be noted that, for very large values of n in equation (42), $\bar{\eta} = 3.6\, n^{-\frac{1}{3}}$. This is to be compared with $\bar{\eta} = 3.902\, n^{-\frac{1}{3}}$ given in equation (35).

The analysis of reference 9 yields the following expression for $P_n'(0)$:

$$P_n'(0) = \frac{1}{\bar{\eta}}\left[\frac{1}{\alpha\bar{\eta}} + \frac{1}{2}n\bar{\eta}^2 - \frac{1}{3\alpha} - \bar{\eta}^2\left(\frac{n}{6} - \frac{1}{72}\right) + \frac{\alpha}{864}\bar{\eta}^5\right] \tag{43}$$

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Figure 5 presents a plot of the values of $P_n'(0)$ calculated from equation (41) and from equation (43). The ten exact values of $P_n'(0)$ obtained in reference 5 are also plotted to show the agreement with the approximate solutions. The better agreement shown by the solution given in equation (41) may be accounted for as follows. First of all, the coefficient of A_n in equation (41) is based on an exact solution of equation (13). This is contrasted with the cubic polynomial representation in the neighborhood of $\eta = 0$ presented in reference 9. However, this must be weighed against the neglecting of the term $FF''/2$ in the second coefficient of equation (12). For small η this approximation is reasonable. Secondly, A_n in equation (41) depends upon the value of $\overline{\eta}$, or equivalently $\overline{\zeta}$; and it might, therefore, be inferred that the value of $P_n'(0)$ would be critically influenced by the solution for large values of η and the corresponding matching of the solutions for large and small η. However, figure 6, which presents a plot of A_n against $\overline{\zeta}$, shows that the value of A_n is relatively insensitive to changes in $\overline{\zeta}$ for a wide range of values. Hence, it would appear that the determination of $P_n'(0)$ in the present analysis is principally affected by the accuracy of the technique used to evaluate the functions in the neighborhood of $\eta = 0$.

Flow deflection at surface. - Assume that the mainstream velocity components are defined by equations (2). Then, if γ designates the local angle of flow deflection at a point on the bounding surface of the flow, it follows from equations (3) and (5) that

$$\begin{aligned} \tan\gamma &= \lim_{y\to 0}\frac{w}{u} \\ &= \lim_{\eta\to 0}\sum_{n=0}^{m}\frac{a_n^* x^n P_n(\eta)}{F'(\eta)} \\ &= \sum_{n=0}^{m}\frac{a_n^* x^n P_n'(0)}{F''(0)} \end{aligned} \tag{44}$$

where $a_n^* = a_n/U_0$ and where L'Hospital's rule is used to evaluate the indeterminate form $P_n(0)/F'(0)$.

As equation (41) has been shown to yield a reasonable approximation to $P_n'(0)$ for all n, equation (44) can be written as

$$\tan\gamma \approx \sum_{n=0}^{m} b_n^* x^n n^{\frac{2}{3}} \tag{45}$$

where

$$b_n^* = 1.354\,\frac{a_n^*}{F''(0)} = 4.08\,a_n^*$$

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For the special case

$$\left.\begin{aligned} W &= a_n x^n \\ U &= U_0 \end{aligned}\right\} \tag{46}$$

equation (45) becomes

$$\tan\gamma = b_n^* x^n n^{\frac{2}{3}} \tag{47}$$

Limiting streamline. - The equation for the "limiting streamline" (line of limiting flow deflection, ref. 5) at the surface can be determined directly from the relation

$$\frac{dz_l}{dx} = \left(\frac{w}{u}\right)_{y=0} \cong \sum_{n=0}^{m} b_n^* x^n n^{\frac{2}{3}} \tag{48}$$

where z_l is the z-coordinate of the limiting streamline.

Solution of equation (48) gives

$$z_l \cong \sum_{n=0}^{m} b_n^* x^{n+1}\,\frac{n^{\frac{2}{3}}}{n+1} + \text{constant} \tag{49}$$

as the approximating equation of the limiting streamline.

For flows defined by equation (46), equation (49) becomes

$$z_l = b_n^* x^{n+1}\left(\frac{n^{\frac{2}{3}}}{n+1}\right) + \text{constant}$$

Now, the main-flow streamlines for this case are given by

$$z = \frac{a_n^* x^{n+1}}{n+1} + \text{constant}$$

If the constant of integration is chosen equal to zero, the ratio of z_l to z is given by

$$\frac{z_l}{z} = \frac{b_n^*}{a_n^*} n^{\frac{2}{3}} = 4.08\ n^{\frac{2}{3}} \tag{50}$$

Hence, the ordinates of the boundary-layer limiting streamline can be determined at once from the ordinates of the main-flow streamline through the origin merely by multiplying the latter by the scale factor $4.08\ n^{\frac{2}{3}}$. The entire system of limiting and main-flow streamlines can then be obtained by translation parallel to the leading edge.

<u>Shear stress at surface.</u> - The shear stress at the bounding surface is given by

$$\tau_0 = \sqrt{(\tau_{0,x})^2 + (\tau_{0,z})^2} \tag{51}$$

where (using eqs. (3) and (5) and the definition of η)

$$\tau_{0,x} = \left(\mu \frac{\partial u}{\partial x}\right)_{y=0} = \mu U_0 F''(0) \sqrt{\frac{U_0}{\nu x}} \tag{52}$$

$$\tau_{0,z} = \left(\mu \frac{\partial w}{\partial y}\right)_{y=0} = \mu \sum_{n=0}^{m} a_n^* x^n p_n'(0) \sqrt{\frac{U_0}{\nu x}} \tag{53}$$

It follows from equations (51), (52), (53), and (41) that τ_0 can be expressed as

$$\tau_0 \cong \frac{\mu U_0^{\frac{3}{2}}}{\sqrt{\nu x}} \sqrt{F''(0) + \left[\sum_{n=0}^{m} a_n^* x^n \left(1.354\ n^{\frac{2}{3}}\right)\right]^2} \tag{54}$$

For flows defined by equation (46), equation (54) becomes

$$\tau_0 \cong \frac{\mu U_0^{\frac{3}{2}}}{\sqrt{\nu x}} n^{\frac{1}{3}} \sqrt{F''(0) n^{-\frac{2}{3}} + (a_n^*)^2 (1.354\ x^n)^2}$$

Therefore, for sufficiently large values of n,

$$\tau_0 \cong \frac{\mu U_0^{\frac{3}{2}}}{\sqrt{\nu x}} a_n^* (1.354) x^n n^{\frac{1}{3}}$$

CONCLUDING REMARKS

A method has been developed for obtaining approximate solutions of the similarity equations for three-dimensional laminar-boundary-layer flows over a flat surface under main-flow streamlines that are translates. The approximate solution makes possible the analysis of flows having streamline shapes representable by polynomials of any degree.

For the particular case of streamline shape described by $z = ax^n$, relatively simple expressions can be found for flow deflection at the surface, limiting streamline shape, and shear stress at the surface.

Comparison of the approximate solution with exact solutions obtained on high-speed computing equipment shows good agreement for values of $n = 10$. In particular, the approximate solution in the vicinity of the plate surface shows relatively good agreement with the exact solution for all values of n.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, November 23, 1956

APPENDIX A

SYMBOLS

A_n	constants of integration
a_n	constants
B_n	constants of integration
b_n	constants
$F,F(\eta)$	Blasius function (eq. (6))
$\mathscr{F}_n,\mathscr{F}_n(\eta)$	function of similarity parameter (eq. (11))
$H_n,H_n(\zeta)$	function of ζ (eq. (14a))
I_ν,K_ν	Bessel functions of imaginary argument (eq. (17))
m,n	constants
$P_n,P_n(\eta)$	function of similarity parameter (eq. (7))
$Q(\zeta)$	function of ζ (eq. (38))
U,W	components of mainstream velocity in x- and z-directions, respectively
U_0	inlet velocity of main stream
u,v,w	components of boundary-layer velocity in x,y,z directions, respectively
x,y,z	rectangular coordinates
α	constant, $= F''(0) = 0.33206$
Γ	gamma function
γ	flow deflection angle at surface
ζ	function of similarity parameter η (eq. (14b))

η	similarity parameter, $\eta = y\sqrt{U_0/\nu x}$
μ	coefficient of absolute viscosity
ν	coefficient of kinematic viscosity
τ_0	shear stress at wall

Subscripts: 4337

c	value of complementary function (eq. (16))
l	limiting
m	constant
n	index number
x,z	x- and z-components, respectively
ν	order of Bessel functions

Superscripts:

'	differentiation
*	division by U (ref. 5)
—	value at matching point

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APPENDIX B

DERIVATION OF EQUATION (18)

Let s_1 and s_2 be solutions of the homogeneous equation obtained from

$$Y'' + p(X)Y' + q(X)Y = r(X) \tag{B1}$$

by setting r equal to zero. Then, reference 18 shows that the general integral of (B1) is

$$Y = -s_1\int\frac{rs_2}{S}\,dX + s_2\int\frac{rs_1}{S}\,dX$$

where $S = s_1s_2' - s_1's_2$.

Applying this technique to equation (15) and using $I_{\frac{1}{3}}$ and $K_{\frac{1}{3}}$ as solutions of (16) give the following solution for H_n:

$$H_n = -n^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}\alpha^{-\frac{4}{3}}\left(-I_{\frac{1}{3}}\int\frac{K_{\frac{1}{3}}\zeta^{-1}}{I_{\frac{1}{3}}K_{\frac{1}{3}}' - I_{\frac{1}{3}}'K_{\frac{1}{3}}}\,d\zeta + K_{\frac{1}{3}}\int\frac{I_{\frac{1}{3}}\zeta^{-1}}{I_{\frac{1}{3}}K_{\frac{1}{3}}' - I_{\frac{1}{3}}'K_{\frac{1}{3}}}\,d\zeta\right) \tag{B2}$$

Reference 16 shows that

$$I_{\frac{1}{3}}K_{\frac{1}{3}}' - I_{\frac{1}{3}}'K_{\frac{1}{3}} = -\zeta^{-1} \tag{B3}$$

Substituting (B3) into (B2) then gives

$$H_n = n^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{4}{3}}\alpha^{-\frac{4}{3}}\left(-I_{\frac{1}{3}}\int K_{\frac{1}{3}}\,d\zeta + K_{\frac{1}{3}}\int I_{\frac{1}{3}}\,d\zeta\right) \tag{B4}$$

By expressing the integrals in equations (B4) as definite integrals with additive constants, equation (18) is obtained.

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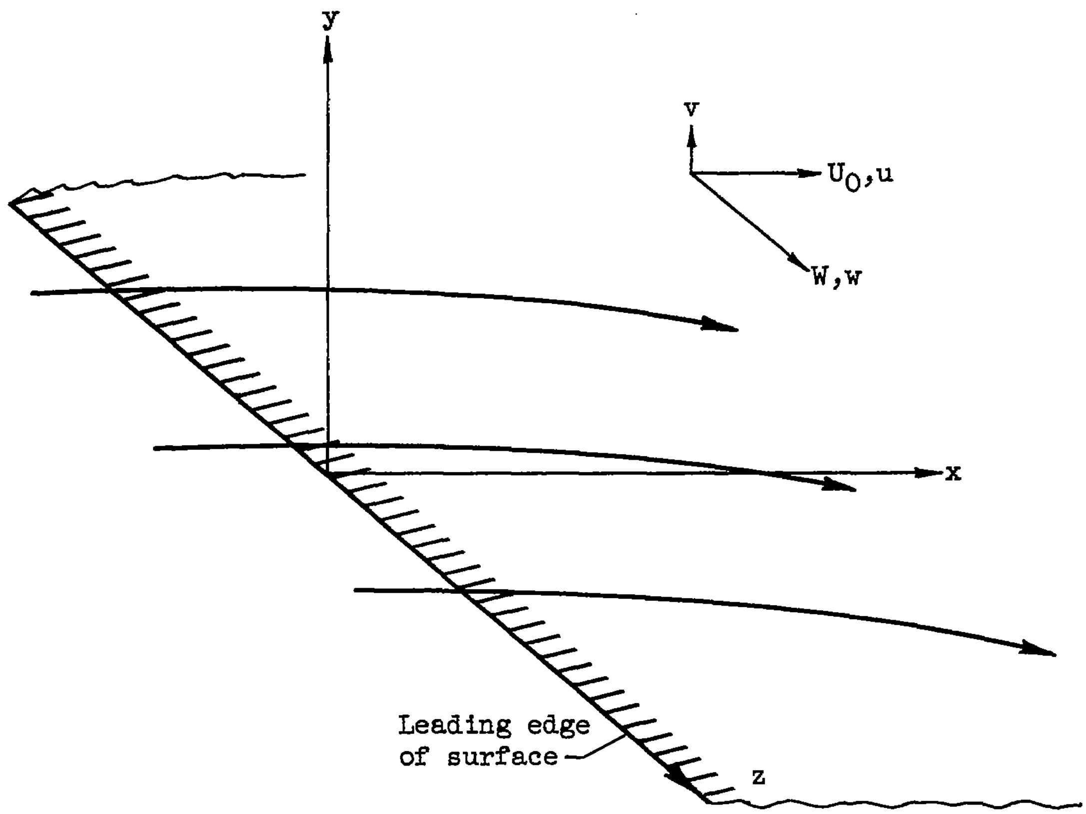

Figure 1. - Coordinate axes for flow over surface.

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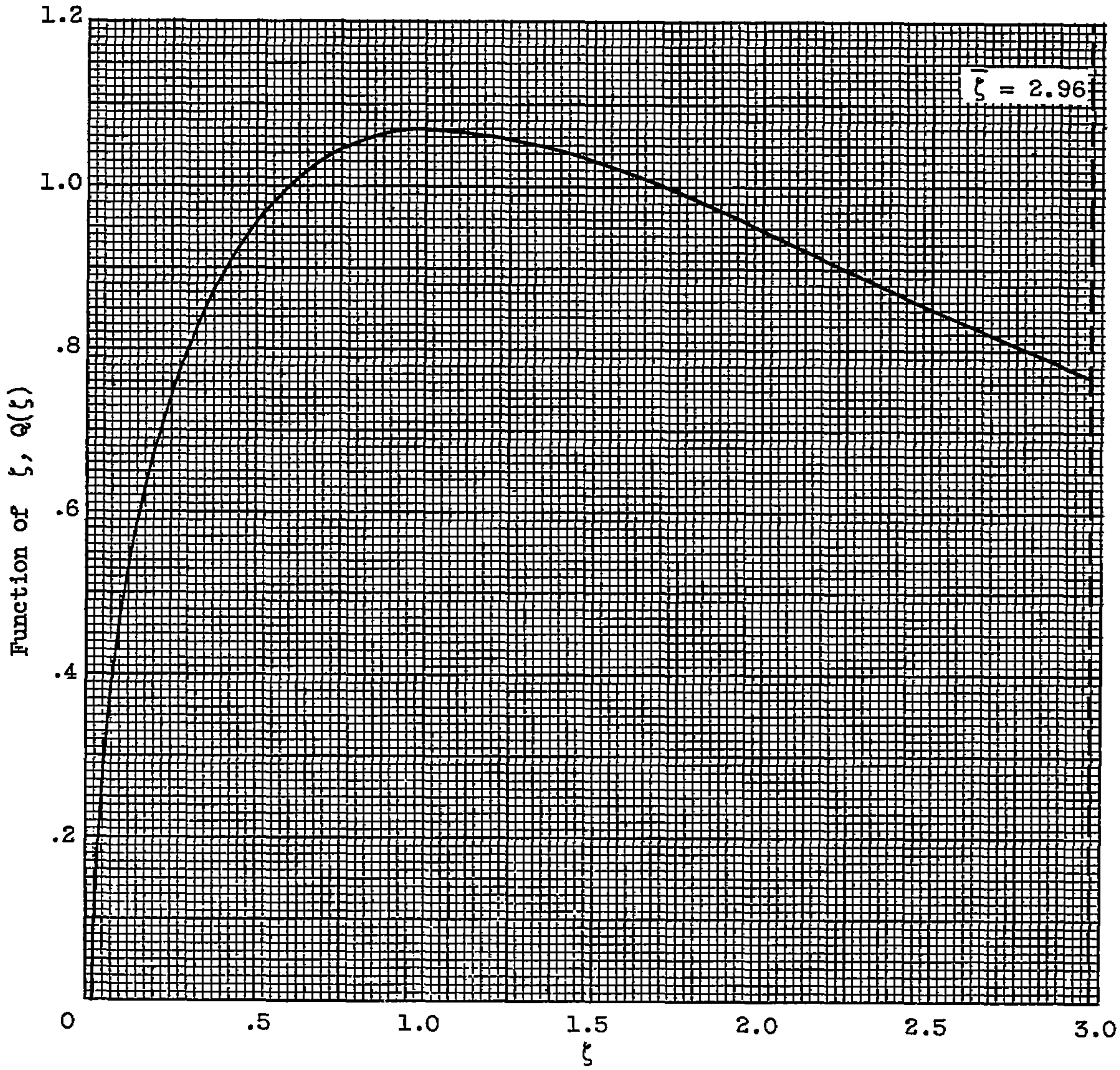

Figure 2. - Variation of function $Q(\zeta)$.

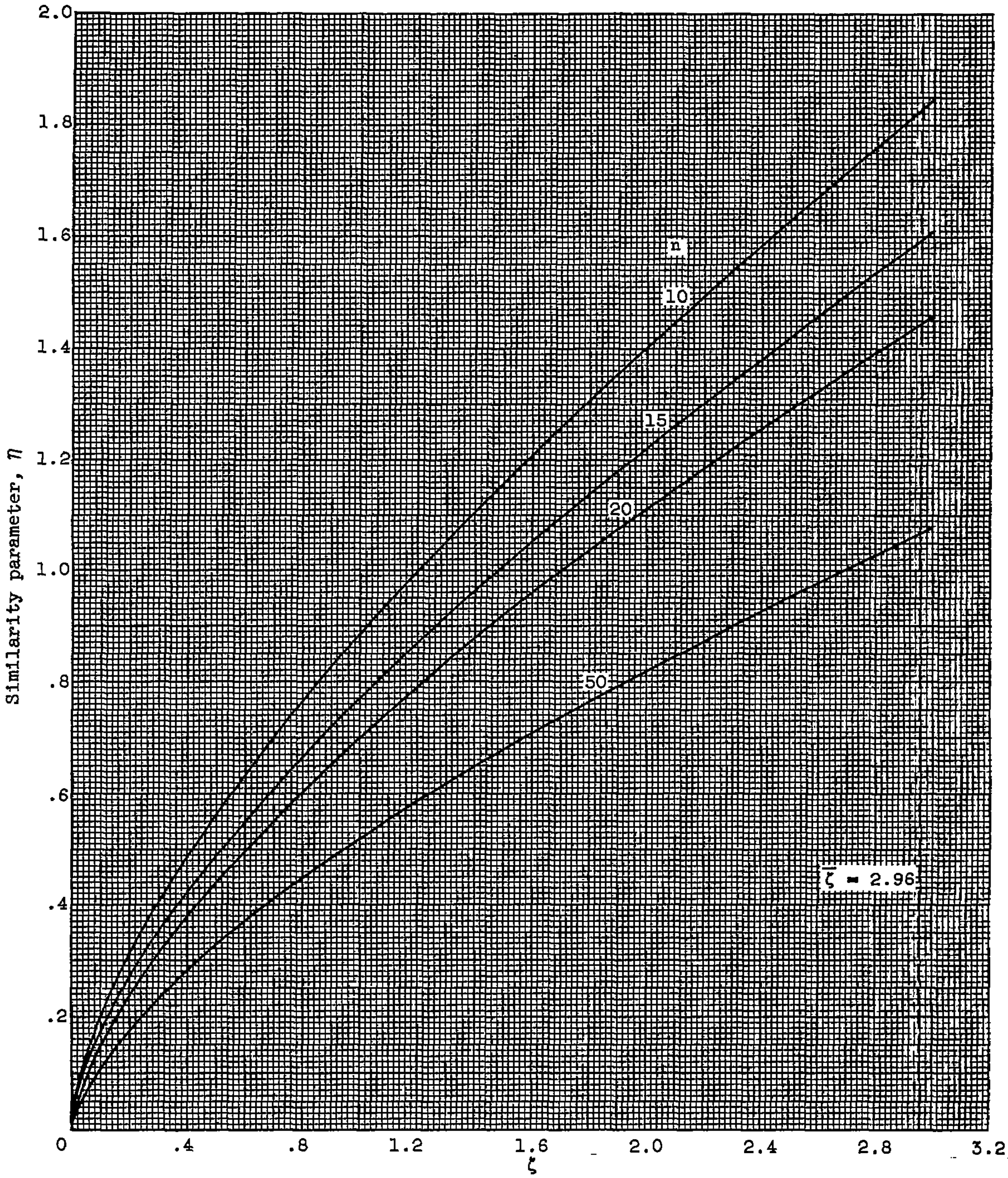

Figure 3. - Variation of similarity parameter η with variable ζ.

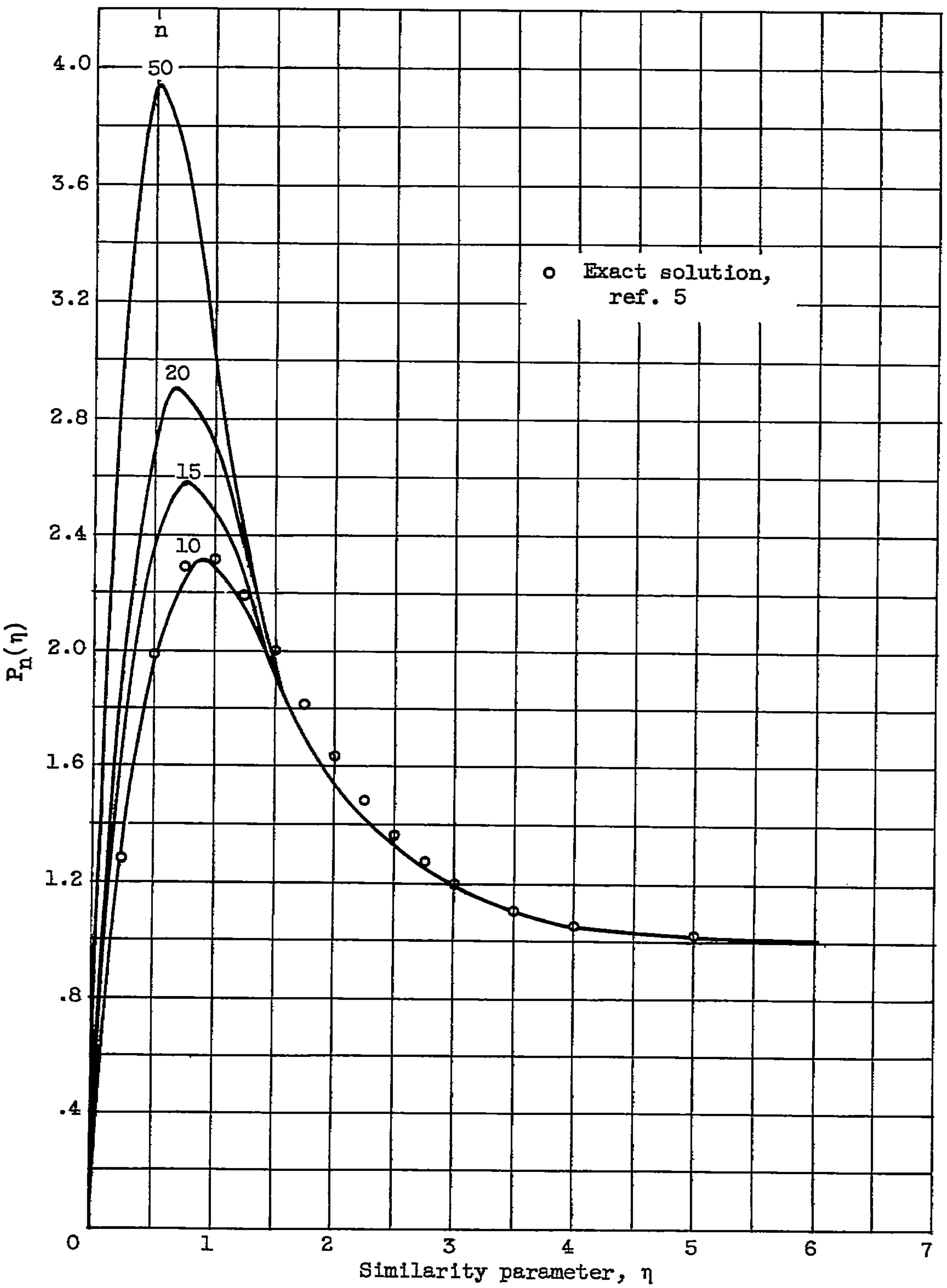

Figure 4. - Approximate solutions of $P_n(\eta)$.

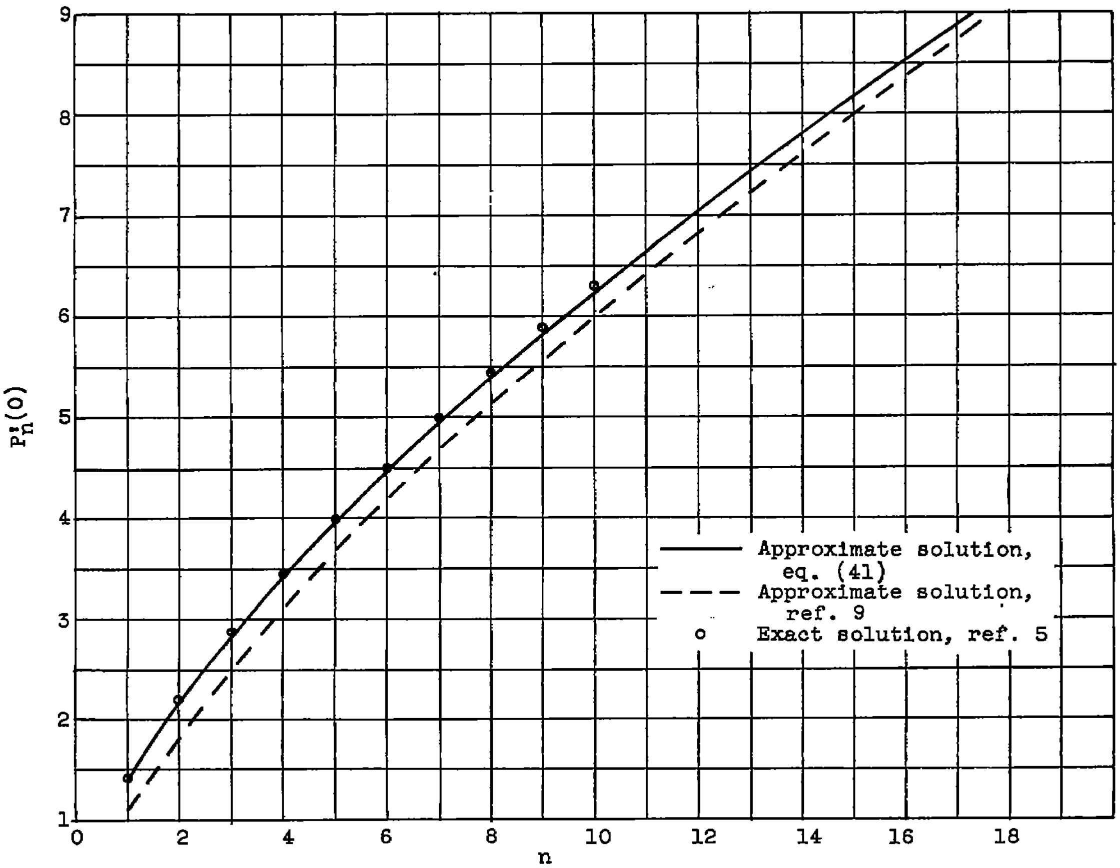

Figure 5. - Comparison of approximate and exact solutions for $P_n'(0)$.

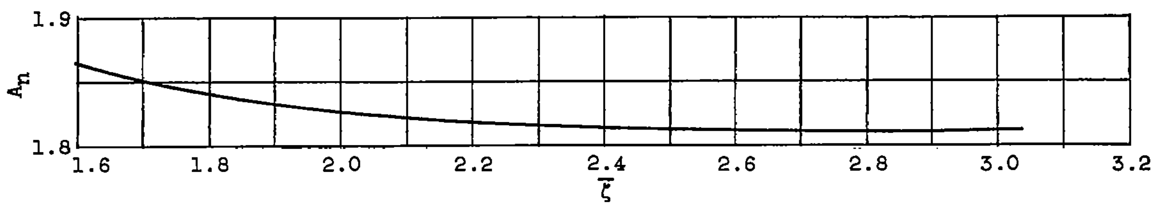

Figure 6. - Variation of A_n with function ζ at matching point.